Int. J. Solids Structures Vol. 27, No. 15, pp. 1915-1924, 1991
Printed in Great Britain.
0020-7683/91 $3.00 + .00

DEFORMATION AND BUCKLING OF AXISYMMETRIC VISCOPLASTIC SHELLS UNDER THERMOMECHANICAL LOADING

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(*Received* 27 *January* 1990; *in revised form* 15 *August* 1990)

Abstract—Viscoplastic material behavior could influence the inelastic collapse and buckling loads of shell structures due to the strain rate dependence of the flow stress and through pre-buckling creep deformations and inelastic unloading. An examination of these effects is performed by incorporating the unified elastic–viscoplastic constitutive equations of Bodner–Parton into the BOSOR-5 computer program of Bushnell for the deformation and buckling of axisymmetric shells subjected to both pressure and thermal loadings. The combined computer program enables consideration of both time-dependent geometrical and material effects on the pre-buckled state and the instability condition. In particular, the procedure enables the determination of a bifurcation-type lower bound on the buckling load. A few numerical exercises were performed to illustrate the various effects, including that of temperature dependence of the viscoplastic properties.

INTRODUCTION

The time dependence of inelastic deformation, manifested by creep, stress relaxation and strain rate sensitivity of the plastic flow stress, is an essential material property which could have a number of direct effects on plastic buckling problems:

(1) pre-buckling creep deformations influence the geometry and the associated stress and strain state so as, in general, to weaken and destabilize the system;
(2) the corresponding strain rate dependence of the flow stress causes the collapse (maximum) load to depend on the loading rate and also influences the bifurcation condition;
(3) inelastic strains would be developed during unloading which could effect the pre-buckled state.

For a structure under constant or slowly increasing loading, creep deformations would generally magnify both the initial imperfections and the load-induced deformations thereby altering the geometry of the system. The geometrical changes could destabilize the system by increasing the moments of the external forces in the case of columns, plates and shells, and by modifying the basic configuration of shallow arches and spherical caps. In conventional formulations, time-dependent creep response is assumed to be non-associated with time-independent elastic–plastic behavior except as the creep deformations influence the geometrical state. As a consequence, elastic buckling could be induced by geometry changes due to creep, e.g. Hoff (1968, 1975), Obrecht (1977), Storåkers (1977) and Hayman (1981). The elastic buckling modes could be due to bifurcation from the original pre-buckled shape. In the inelastic range, the material stiffness would decrease due to plastic strains developed by the increased bending moments. With the conventional procedure, these plastic strains are obtained from a time-independent inelastic analysis based on the current (time-dependent) geometry of the system. This general approach to inelastic buckling problems in the presence of creep deformations is well represented in the literature, e.g. Bushnell (1974).

Apart from the more obvious effect of creep deformations on destabilization, the strain rate dependence of the flow stress influences possible buckling conditions, i.e. collapse or bifurcation, in a complicated manner. In the pre-buckled state, longer times of loading would generally correspond to lower strain rates and would thereby reduce the effective material stiffness in the inelastic range. The collapse condition determined by a maximum of the load deflection relation would therefore be a function of loading rate, or time, for a structure of viscoplastic material, e.g. Tvergaard (1985).

For the bifurcation condition, the role of strain rate sensitivity of the flow stress is more complex. Rabotnov and Shesterikov (1957) (referred to as R–S), Gerard and Papirno (1962), Gerard (1962) and Trojnacki and Zyczkowski (1976), among others, treated creep buckling as a bifurcation problem by proposing a load- and time-dependent degradation of material stiffness. The investigation of R–S was extensively discussed by Hoff (1958), and by Jahsman and Field (1962a,b), who suggested alterations in the reference equation of state, e.g. consideration of elastic unloading and the creep hesitation effects upon reduction of load. Many commentators on the problem at that time (in the 1960s and 1970s) pointed to the inadequacies of the currently available equations of state for the material response under the required conditions. Another evaluation of the R–S approach by Yamamoto (1970) criticized the use of a dynamic stability criterion for creep buckling of columns. More recently, Tvergaard (1989), as well as other investigators, have shown that Hill's (rate-independent) bifurcation criterion can be satisfied for a general elastic–viscoplastic structure only at the elastic buckling load. This comes about since the incremental response of a viscoplastic material to an instantaneous change of strain rate is elastic. The result is, of course, singular and practically unrealistic since imperfections and inertial effects would nullify the assumption of an instantaneous strain rate change. An alternative approach is to establish the effective tangent modulus of the material for the existing strain rates in the pre-buckled deformation state and to use a suitably integrated (over the structure) tangent modulus in the conventional buckling criterion. With an appropriate constitutive theory, the tangent modulus at each point in the structure could be obtained as the coefficient in an expression for the incremental stress–strain relation at the current state condition which would be compatible with Hill's bifurcation condition. The resulting bifurcation-type buckling load could then be interpreted as a lower bound on the actual buckling condition since higher strain rates would be realized at the initiation of the buckling process. Further discussion of this aspect of the buckling problem will be the subject of a subsequent paper. The present one is primarily concerned with describing the coupling of the elastic–viscoplastic theory of Bodner–Partom (referred to as B–P) to the BOSOR-5 program and the presentation of some numerical results for the buckling of cylindrical shell structures subjected to both external pressure and a temperature gradient through the shell wall. The B–P formulation enables direct consideration of the temperature dependence of the viscoplastic properties within the BOSOR-5 program.

ELASTIC–VISCOPLASTIC CONSTITUTIVE EQUATIONS

In recent years, more realistic constitutive euqations have been proposed for the combined elastic and time-dependent inelastic deformation of metals. One class of such equations is referred to as "unified" in the sense that all inelastic deformations are represented by the same variable. As a consequence, strain rate-dependent plastic flow, creep and stress relaxation are consequences of particular loading conditions for the same set of governing equations. These equations also include load history-dependent variables to represent certain aspects of the material state, e.g. hardening. Some of the proposed sets of "unified" constitutive equations do not require a prescribed yield criterion and loading/unloading conditions. The particular constitutive equations of Bodner and Partom (1972, 1975) for elastic–viscoplastic response are of this class and are adopted in the present paper. These were initially proposed in 1968, and their current version was reviewed by Bodner (1985, 1987). These constitutive equations have been shown to realistically represent material response characteristics over a wide range of loading and temperature conditions, e.g. Chan *et al.* (1988, 1989, 1990). The applications demonstrate the representational and predictive capability of the equations with respect to the strain rate dependence of the flow stress, creep under steady and varying loads, stress relaxation under uniaxial and multiaxial loading conditions, and uniaxial and multiaxial cyclic behavior.

Use of such constitutive equations in buckling problems enables simultaneous consideration of both time-dependent geometrical and material effects and provides an overall

Table 1. A summary of the elastic–viscoplastic constitutive model

Decomposition of strain rate:

$$\dot{\varepsilon}_{ij} = \dot{\varepsilon}^{e}_{ij} + \dot{\varepsilon}^{p}_{ij} \tag{1}$$

Flow law:

$$\dot{\varepsilon}^{p}_{ij} = \lambda s_{ij}; \quad \dot{\varepsilon}^{p}_{kk} = 0 \tag{2}$$

with $s_{ij} = \sigma_{ij} - (1/3)\delta_{ij}\sigma_{kk}$

Kinetic equation:

$$D^{p}_{2} = D^{2}_{0} \exp\left[-\left(\frac{Z^{2}}{3J_{2}}\right)^{n}\right] \tag{3}$$

with $D^{p}_{2} = (1/2)\dot{\varepsilon}^{p}_{ij}\dot{\varepsilon}^{p}_{ij}$; $J_{2} = (1/2)s_{ij}s_{ij}$

$$\lambda^{2} = D^{p}_{2}/J_{2} \tag{4}$$

$$Z = Z^{I} + Z^{D} \tag{5}$$

Evolution equations of internal variables:

(a) Isotropic hardening

$$\dot{Z}^{I} = m_{1}[Z_{1} - Z^{I}]\dot{W}_{p} - A_{1}Z_{1}\left[\frac{Z^{I} - Z_{2}}{Z_{1}}\right]^{r_{1}} \tag{6}$$

with $Z^{I}(0) = Z_{0}$; $\dot{W}_{p} = \sigma_{ij}\dot{\varepsilon}^{p}_{ij} = \sigma_{eff}\dot{\varepsilon}^{p}_{eff}$; $W_{p}(0) = 0$

(b) Directional hardening

$$\dot{\beta}_{ij} = m_{2}(Z_{3}u_{ij} - \beta_{ij})\dot{W}_{p} - A_{2}Z_{1}\left[\frac{(\beta_{kl}\beta_{kl})^{1/2}}{Z_{1}}\right]^{r_{2}} v_{ij} \tag{7}$$

where $u_{ij} = \sigma_{ij}/(\sigma_{kl}\sigma_{kl})^{1/2}$, $v_{ij} = \beta_{ij}/(\beta_{kl}\beta_{kl})^{1/2}$

$$\text{and} \quad Z^{D} = \beta_{ij}u_{ij}; \quad Z^{D}(0) = 0, \quad \beta_{ij}(0) = 0 \tag{8}$$

Material constants:

D_{0}, n, Z_{0}, Z_{1}, Z_{2}, Z_{3}, m_{1}, m_{2}, A_{1}, A_{2}, r_{1}, r_{2}, and elastic constants; with $n = n(T)$ (in most cases one can set $Z_{2} = Z_{0}$, $A_{1} = A_{2}$, $r_{1} = r_{2}$)

framework for the treatment of problems involving variable loading histories such as uniform loading rates, steady loading (creep buckling), impulsive loading and repeated loading. Combined thermal and mechanical loading conditions could also be considered within such a formulation which can include temperature dependence of the viscoplastic properties.

The B–P elastic–viscoplastic equations are given in a slightly specialized form in Table 1. In the decomposition of the total strain rate into elastic and inelastic components, eqn (1), it is noted that both components are generally non-zero for all conditions of loading and unloading. This means that plastic strains, although very small, are present at low stress levels and upon unloading. The proposed flow law, eqn (2), is that associated with the von Mises yield criterion although yield and normality conditions are not required by the material model. That law is then a relation between the direction of the physical plastic strain rate and the deviatoric stress. By means of the kinetic eqn (3), which relates the invariants of plastic strain rate D^{p}_{2} and deviatoric stress J_{2}, the coefficient of the flow law λ can be determined from eqn (4) as a function of stress and the load history-dependent hardening variable Z. That variable could be interpreted as a measure of the resistance to plastic flow.

Another term appearing in the kinetic equation is the coefficient D_{0} which corresponds to the limiting plastic strain rate in shear. This interpretation is inherent in the functional form of eqn (3), and an assumed value for D_{0} can be used which is less than the presumed physical one. The other parameter is n which controls strain rate sensitivity and the overall level of the flow stress; it is temperature- and pressure-dependent with n, in general, varying inversely with temperature and directly with superimposed hydrostatic pressure. Lower values of n correspond to increased strain rate sensitivity and reduced levels of the flow stress.

The earlier formulation of the constitutive equations (Bodner and Partom, 1975) considered only isotropic hardening but both isotropic and directional hardening effects are contained in the more recent version (Bodner, 1985, 1987). In the B–P equations, the total scalar hardening variable Z is considered to be composed of isotropic and directional components Z^{I} and Z^{D} respectively, eqn (5). Evolution equations for those quantities are of saturation form, eqns (6) and (7), and include terms corresponding to thermal recovery of hardening which enable the condition of secondary creep to develop. The material constants associated with thermal recovery of hardening are Z_2, A_1, A_2, r_1 and r_2. Directional hardening is actually represented as a second-order tensor, β_{ij}, and its evolution eqn (7) is also tensorial. A scalar effective value, Z^{D}, is taken to be the component of β_{ij} in the direction of the current stress, eqn (8), and is added to the isotropic hardening Z^{I} to make up the total hardening variable Z.

Directional hardening effects could enter into buckling problems due to the activation of stress components upon buckling that are zero in the pre-buckled state. Those effects, however, would be small in the applications under discussion. Inclusion of directional hardening is, nevertheless, desirable since the ability of the equations to properly model actual material behavior is considerably improved. The equations in Table 1 do not include the possible additional hardening effects due to extended non-proportional loading histories (Bodner, 1987), which are generally not significant in buckling problems.

SUMMARY OF THE BOSOR-5 PROGRAM

The BOSOR-5 computer program of Bushnell (1973, 1974, 1976a,b, 1985) determines the elastic plastic stresses and deformations of shells of revolution subjected to axisymmetrical loading, and also the load condition for plastic collapse or for bifurcation buckling. It is a finite difference program that uses the principle of virtual work to establish the equilibrium conditions where the nodal point movements serve as the virtual displacements. Since rotational symmetry is assumed prior to buckling, only the circumferential and meridional pre-buckling stress components are non-zero in the pre-buckled state. Also, only axially symmetrical initial imperfections can be included. Hooke's Law is taken for the elastic stress–strain relation and thermal, strain rate-independent plastic and time-dependent creep strains are included. The basic equilibrium equations are integrated throughout the shell thickness and over the shell surface to give a set of non-linear algebraic equations. These are solved numerically by the Newton–Raphson method which involves satisfaction of an iteration equation for the increments of the nodal point values; see eqn (3) of Bushnell (1974).

For inelastic material behavior, time-independent incremental and deformation plasticity theories have been incorporated within the basic BOSOR-5 program. To provide a basis for generalization to rate-dependent plasticity, the J_2 incremental plasticity theory with isotropic hardening was chosen as the reference. The tangent modulus procedure is used in the program which involves the determination of a plastic loading matrix and a tangent stiffness matrix which relates increments of stress and strain. In the standard method, the tangent stiffness matrix is independent of the creep and thermal strains. This means that those strains would have no direct influence on the plastic response but would be expected to effect it indirectly through the changes in the geometry of the structure.

Plastic collapse (instability) of the shell in the axisymmetrical mode would be indicated by failure of the procedure to converge at some level of applied pressure. For bifurcation buckling in a symmetrical or non-symmetrical mode, a standard bifurcation analysis based on an integrated tangent modulus is adopted. Since the pre-buckled state is asixymmetric, only bifurcation buckling is possible for a non-symmetrical mode. The possible mode patterns would generally involve shear strains, which would require consideration of additional degrees of freedom in the nodal displacements. Bifurcation buckling would be indicated by a determinant obtained from the variation of the equilibrium conditions becoming zero. In practice, the determinant is evaluated at each load step and the load step at which the determinant change sign gives the range in which buckling would take place. A detailed discussion of the buckling analysis appears in Bushnell (1974).

INCLUSION OF THE B–P ELASTIC-VISCOPLASTIC EQUATIONS IN THE BOSOR-5 PROGRAM

Within the context of the J_2 incremental plasticity formulation of the BOSOR-5 computer program, the use of the B–P constitutive model requires modification of a number of the basic equations and of the numerical procedure for handling the essential time dependence. An important change is that plasticity and creep are not uncoupled in the unified B–P theory and are represented by the same variable. A single inelastic strain term therefore appears instead of the sum of the uncoupled plastic and creep strains in the expression for the strain energy of the shell elements. Similarly, the expression for the inelastic strain in terms of the total and thermal strains includes both plasticity and creep, and depends on the nodal deformations.

Because of the inherent time dependence and the coupling of plasticity and creep, it is necessary to recognize that the inelastic tangent modulus, E_T^p, which appears in the BOSOR-5 formulation, cannot be deduced from the response characteristics. That is, E_T^p should be a property of the material state (Z) and the applied stress, while the time-dependent inelastic response and the changes in stress depend upon the prevailing loading conditions. In practice, this means that E_T^p should be obtained at each time increment from the reference constitutive equations for the current values of stress and hardness (Z) and not from the derived increments of stress and inelastic strain. This may be part of the reason for the difficulties experienced by the creep buckling theories of R–S and of Gerard in which the respective material stiffnesses are obtained from the response characteristics. The problem does not arise for the uncoupled plasticity–creep formulations in which creep only influences the structural geometry while E_T^p is directly related to the ratio of the increments of stress and rate-independent plastic strain.

For the B–P constitutive theory, Table 1, it is possible to obtain an explicit expression for the short time inelastic tangent modulus of the pre-buckled state directly from the equations which is then used in the equilibrium equations and in the determinant for bifurcation buckling. As discussed previously, this bifurcation buckling value should be a lower bound on the actual buckling condition, since higher strain rates would be realized during the buckling process. Taking $\sigma_{\text{eff}} = \sqrt{3J_2}$, the kinetic eqn (3) can be expressed as

$$\sigma_{\text{eff}}/Z = \text{function of } (D_2^p/D_0^2) = f(D_2^p/D_0^2) \tag{9}$$

where D_2^p is the second invariant of the plastic strain rate in the pre-buckled state. The inelastic tangent modulus corresponding to an increment of effective inelastic strain, $d\varepsilon_{\text{eff}}^p$, with the plastic strain rate held steady, would then be

$$E_T^p = \frac{d\sigma_{\text{eff}}}{d\varepsilon_{\text{eff}}^p} = \frac{dZ}{d\varepsilon_{\text{eff}}^p} f(D_2^p/D_0^2). \tag{10}$$

It is noted that eqn (10) would reduce to the standard tangent modulus in the limit of a rate-independent solid. The term $(dZ/d\varepsilon_{\text{eff}})$ can be obtained from the evolution eqns (6) and (7), Table 1, neglecting thermal recovery terms for the short time behavior. Multiplying eqn (6) by $(dt/d\varepsilon_{\text{eff}}^p)$ and eqn (7) by $u_{ij}(dt/d\varepsilon_{\text{eff}}^p)$ and adding, leads to

$$E_T^p = [m_1(Z_1 - Z^I) + m_2(Z_3 - Z^D)]\sigma_{\text{eff}} f(D_2^p/D_0^2). \tag{11}$$

This expression gives the inelastic tangent modulus of the pre-buckled state in terms of the current state quantities Z^I, Z^D and the current inelastic strain rate invariant D_2^p which is assumed to be steady at the condition of obtaining E_T^p. Alternatively, using eqn (9), E_T^p can be expressed in terms of the current stress and the state quantities,

$$E_T^p = [m_1(Z_1 - Z^I) + m_2(Z_3 - Z^D)][\sigma_{eff}^2/(Z^I + Z^D)]. \tag{12}$$

This equation was used for the term H' in the BOSOR-5 program which appears in the plastic loading matrix $[C]$; see eqn (6) of Bushnell (1974). That matrix occurs in the terms that arise in the iteration procedure for satisfying the equilibrium equations and also in the determinant for the bifurcation condition. The short time plastic tangent modulus therefore has a direct influence on the bifurcation condition and a minor effect on the determination of the pre-buckled state. It is noted that although eqn (12) does not involve the thermal recovery terms, those terms do influence E_T^p through the current values of the hardening variables Z^I and Z^D which are based on the full eqns (6) and (7). Identification of the pre-buckled state requires satisfaction of the complete set of equilibrium and constitutive equations and the boundary conditions for the given problem.

For the numerical solution of the governing elastic–viscoplastic material equations at each increment of time, it is convenient to use the implicit numerical scheme developed by Kanchi *et al.* (1978). The B–P model can be used directly to obtain analytical expressions between the increments of inelastic strain rate and stress. Also, as discussed above, the plastic loading matrix $[C]$ is obtained from the relation between the inelastic strain and stress increments and from the expression for the stress increment in terms of the elastic strain increment. It is noted that with the B–P model, $[C]$ is influenced by both plasticity and creep and is always non-zero. Alternatively, in the classical yield surface plasticity formulation, $[C]$ is zero at stress levels below yield and upon unloading. The complete system of equations is solved at each time increment in the basic BOSOR-5 computer program. Details of the procedure are given by Naveh (1987).

In the BOSOR-5 program, the stability of equilibrium is examined at each time increment by evaluation of a determinant based on the second variation of the total energy. For the inherently non-conservative system, this corresponds to examining the condition for neutrality of the equilibrium state from the viewpoint of virtual work. In this investigation of possible bifurcation of the equilibrium state, non-symmetrical modes are also considered, i.e. additional degrees of freedom are introduced. The inelastic tangent modulus obtained analytically from the B–P equations for the current material state, eqn (12), is used as the material stiffness parameter in the resulting stability determinant. The total tangent modulus which relates the increment of stress to the increment of total strain can be readily obtained in terms of E_T^p and the elastic modulus E.

NUMERICAL EXAMPLES

A number of numerical exercises had been carried out with an earlier version of the modified BOSOR-5 program by Bodner and Naveh (1988) to examine the effect of material rate sensitivity on the buckling of stiffened cylindrical shells hydrostatically loaded at a controlled rate. In that study, the plastic tangent modulus E_T^p was evaluated from the response characteristics which overemphasized the influence of material rate sensitivity on the material stiffness for short time structural deformations. However, the numerical results obtained for the relatively rate-insensitive aluminum alloy 7075-T6 agreed well with corresponding test results. Also, various numerical results obtained with the aluminum alloy material were in agreement with those of the original rate-independent plasticity formulation, which used a reference stress–strain curve corresponding to the viscoplastic response at the average developed strain rate. On that basis, the details of the numerical procedure and the programming are considered to be correct.

The program was subsequently further modified with E_T^p evaluated according to the equations given in this paper and this revised program was the basis of the further numerical exercises. The revised program was also generalized to consider thermal strains which are included in the original BOSOR-5 formulation. To expand the treatment of thermal effects, the full evolution equations for the hardening variables with the thermal recovery terms, eqns (6) and (7), were incorporated into the revised program. This enables the condition of secondary creep to develop under constant load conditions. A further generalization was to permit the rate-dependent inelastic properties to be functions of temperature.

For the numerical exercises based on the current version of the computer program, the reference structure was taken to be a closed cylindrical shell subjected to exernal hydrostatic pressure and a linear temperature gradient through the wall thickness that was uniform over the cylindrical surface. The aluminum alloy 7075-T6 defined by the B–P model in the 1988 paper was taken as the shell material. The dimensions of the shell and the B–P material constants are indicated in the Appendix along with an initial set of boundary conditions (set I) which permits uniform radial motion to avoid bending in the pre-buckled state. A series of exercises was undertaken to validate the program, one of which was to check whether a linear temperature gradient through the thickness would have any influence on the elastic buckling pressure. As expected, the numerical results indicated no effect of a linear temperature gradient on the elastic buckling load obtained by setting the yield stress to a high value. In the inelastic range, however, a linear temperature gradient through the shell wall does influence the buckling pressure due to the reduced stiffness of some of the material. In the particular exercise that was performed, a linear temperature gradient of 112 C reduced the buckling pressure by 13%. In that example, buckling was due to bifurcation in a non-symmetrical mode with six circumferential waves.

Further generalization of the modified computer program involved inclusion of the hardening evolution equations in incremental form with terms corresponding to thermal recovery of hardening, eqns (6) and (7). Thermal dependence of the inelastic plastic flow properties can be considered by taking some of the material constants to be functions of temperature. Recent exercises in modeling materials at various temperatures have shown that the parameter n is primarily responsible for temperature effects such as the changes in flow stress and level and in strain rate sensitivity. In general, it is expected that n would decrease with temperature and would thereby lower the flow stress level and increase strain rate sensitivity. An empirical form of the temperature dependence of n used in previous studies is

$$n = (A/T) + B. \tag{13}$$

A series of numerical exercises was conducted in which both thermal recovery of hardening and temperature dependence of n were included. The basic problem for examination was that of the reference shell subjected to a linear temperature gradient through the thickness of 100 C which was uniform over the surface, and to a hydrostatic pressure. The boundary conditions for this problem are given by set II in the Appendix which corresponds to clamped ends. The value of $n = 5$ for the aluminum alloy material was taken to apply at 0 C and other choices of n were assumed to apply at 100 C. Values of A and B were then set to provide the necessary transition, and reasonable values for the thermal recovery constants were chosen. Since the dimensions of the reference geometry and the material constants were arbitrarily chosen, the absolute buckling values are unimportant and only relative percentage changes for different conditions are of interest. Taking the case of n to be temperature-independent as the reference, so that n would also equal 5 at 100 C, it was found that with $n = 3$ at 100 C, the buckling pressure reduced by 12% and with $n = 1$ at 100 C, the buckling pressure reduced by 31%. Again, buckling in all cases was by non-symmetrical bifurcation with six circumferential waves. Temperature dependence of the inelastic material properties could therefore be a significant factor in buckling problems involving thermal effects.

It is noted that the original BOSOR-5 program is not arranged to consider temperature dependence of the elastic modulus or of the inelastic properties. This limitation can be overcome, at least for the inelastic properties, by the use of a suitable temperature-dependent viscoplastic material model.

Another exercise was to examine the effect of loading rate on the buckling pressure for a rate-sensitive material using the equation for E_T^p given in this paper. The same reference shell structure of the Appendix was used and the material was artificial with a rate sensitivity like titanium, $n = 1$, but with higher values of the hardening constants (see Appendix). The boundary conditions corresponded to set II which apply for fully clamped ends and are more realistic. Thermal effects were not considered in this exercise. For pressure loading at a rate corresponding to an average strain rate of 10^{-3} s^{-1}, the computed buckling value (by bifurcation) is taken to be unity for reference. In comparison, the fully elastic buckling pressure for this geometry is 1.09. At a lower loading rate corresponding to an average strain rate of 10^{-6} s^{-1}, the inelastic buckling load (also by bifurcation) attained, using the present formulation, is 0.96. In comparison, the earlier formulation based on obtaining E_T^p from the response results led to an appreciably larger percentage decrease in buckling pressure at the lower loading rate which appeared to overemphasize the effect of viscoplasticity. For that exericse, the material model was based on actual titanium with $n = 1$ and lower values of the hardening parameters. Also, the reference structure for that exercise was different than in the present case. The fairly small 4% reduction in buckling load due to the three decade change in loading rate obtained in the present case may be due, in part, to the inelastic buckling pressure being close to the fully elastic condition for this particular example.

CONCLUSIONS

The B–P elastic–viscoplastic constitutive equations have been implemented into the BOSOR-5 computer program of Bushnell for the deformation and buckling of axisymmetrical shells subjected to both mechanical and thermal loading. The constitutive theory is considered to be "unified" in the sense that plasticity and creep are inherently coupled and are represented by the same variable. Thermal recovery effects and thermal dependence of the inelastic properties are included in the formulation.

As part of the implementation, an analytical expression for the inelastic tangent modulus in the pre-buckled state was obtained directly from the constitutive theory. Use of this modulus in the buckling determinant (bifurcation criterion) leads to a lower bound on the instability load of structures of strain rate-dependent material.

Particular advantages of using a "unified" elastic–viscoplastic constitutive theory in a structural computer program are:

(1) the rate sensitivity of inelastic straining can be taken into account directly at each point in the structure; the rate sensitivity influences the structural response and the buckling and collapse loads;
(2) possible creep of the structure in the pre-buckled state is inherently included;
(3) thermal effects, namely: thermal strains, recovery of hardening, and temperature dependence of inelastic properties, can be accounted for at each point in the structure.

A number of numerical exercises were performed to examine the response of cylindrical shells of elastic–viscoplastic materials subjected to combined thermomechanical loading. With the implemented viscoplastic theory, overall computer running times were generally less than those of the original program (with rate-independent plasticity) for the exercises that were performed. In some cases, the running time for the viscoplastic formulation was about half that for the basic program. This seems to be due to the fairly direct procedure that has been introduced in the modified program compared to the need to continually satisfy the current yield criterion and loading unloading conditions in the original formulation.

Acknowledgements—The research reported in this paper was supported in part by the U.S. Air Force Office of Scientific Research through the European Office of Aerospace Research and Development, London, under Grant AFOSR-87-0391. The authors would also like to thank Mrs Ronit Yaffe of the Aerospace Engineering Department of the Technion for her assistance in the operation of the BOSOR-5 program.

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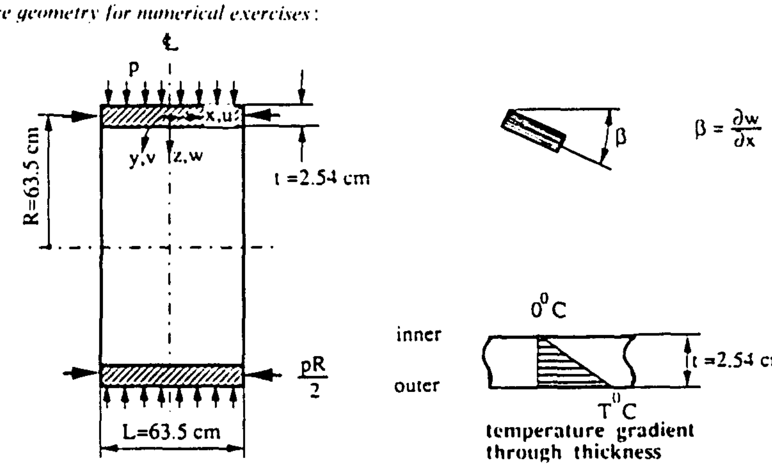
APPENDIX

(1) *Reference geometry for numerical exercises*:



(2) *Boundary conditions for numerical exercises*:

	At ends, $x = \pm L/2$	At centerline, $x = 0$ (symmetry condition)
set I	pre-buckling/for buckling	pre-buckling/for buckling
u	free/free	fixed/fixed
v	fixed/fixed	free/free
w	free/free	free/free
β	fixed/fixed	fixed/fixed
set II	pre-buckling/for buckling	pre-buckling/for buckling
u	free/free	fixed/fixed
v	fixed/free	free/free
w	fixed/fixed	free/free
β	fixed/fixed	fixed/fixed

(3) *Material constants: aluminum alloy* 7075-T6

elastic:

$E = 74.4$ GPa
$\nu = 0.3$

coefficient of thermal expansion:

$\alpha = 2.4 \times 10^{-5}\ {}^{\circ}C^{-1}$

viscoplastic (B/P):

$D_0 = 10^4\ s^{-1}$ (assumed)
$n = 5.0$
$Z_0 = Z_2 = 682$ MPa
$Z_1 = 869$ MPa
$Z_3 = 90$ MPa
$m_1 = 0.087\ (MPa)^{-1}$
$m_2 = 3.23\ (MPa)^{-1}$
$\left.\begin{array}{l} A_1 = A_2 = 0.2\ s^{-1} \\ r_1 = r_2 = 1 \end{array}\right\}$ (arbitrarily chosen)

(4) *Material constants: artificial "titanium alloy"*

elastic:

$E = 118$ GPa
$\nu = 0.3$

viscoplastic (B/P):

$D_0 = 10^4\ s^{-1}$ (assumed)
$n = 1.0$
$Z_0 = 5320$ MPa
$Z_1 = 6532$ MPa
$Z_3 = 1496$ MPa
$m_1 = 0.089\ (MPa)^{-1}$
$m_2 = 2.80\ (MPa)^{-1}$